

Comment on “Quantum Raychaudhuri equation”

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Abstract

We address the validity of the formalism and results presented in [S. Das, Phys. Rev. **D89** 084068 (2014)] with regard to quantum Raychaudhuri equation. The author obtained the so called quantum Raychaudhuri equation by replacing classical geodesics with quantal trajectories arising from Bohmian mechanics. The resulting modified equation was used to draw some conclusions about the inevitability of focussing and the formation of conjugate points and therefore singularity. We show that the whole procedure is full of problematic points, on both physical relevancy and mathematical correctness. In particular, we illustrate the problems associated with the technical derivation of the so called quantum Raychaudhuri equation, as well as its invalid physical implications.

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The central finding in [1] can be summarized as follows: Instead of considering geodesics curves one uses modified curves arising from Klein-Gordon-type equation of the following form (complex scalar field coupled non minimally to gravity),

$$\left(\square - \frac{m^2 c^2}{\hbar} + \epsilon R \right) \Phi = 0, \quad \square \equiv g^{\mu\nu} D_\mu D_\nu. \quad (1)$$

The polar decomposition of the scalar field Φ is utilized as $\Phi = \mathcal{R} \exp\left(\frac{iS}{\hbar}\right)$ and then using Bhom’s interpretation to define new congruence with tangent vector field given by

$$u^\mu = \frac{dx^\mu}{d\lambda} = \frac{D^\mu S}{m}, \quad (2)$$

and consequently the norm of the velocity field u^μ is given by

$$u^\mu u_\mu = -c^2 + \frac{\hbar^2}{m^2} \frac{\square \mathcal{R}}{\mathcal{R}} + \frac{\hbar^2}{m^2} \epsilon R, \quad (3)$$

form which the acceleration field follows as

$$a^\mu \equiv u^\rho D_\rho u^\mu = \frac{\hbar^2}{2m^2} \left\{ D_\mu \left(\frac{\square \mathcal{R}}{\mathcal{R}} \right) + \epsilon D_\mu R \right\}. \quad (4)$$

If we compare this resultant acceleration with the corresponding one given in [1], we find a missing factor 1/2.

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In deriving the quantum version of Raychaudhuri equation (QRE) associated to the Bohmian trajectories, an essentially identical formalism devised for congruence generated by geodesics parameterized with the proper time was used in [1] to define the “spatial” metric as $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and its corresponding projector $h^\mu{}_\nu$, the expansion scalar $\theta = h^{\mu\nu} D_\nu u_\mu$, the shear tensor $\sigma_{\mu\nu} = \frac{1}{2} (D_\mu u_\nu + D_\nu u_\mu) - \frac{1}{3} \theta h_{\mu\nu}$, the rotation tensor $\omega_{\mu\nu} = \frac{1}{2} (D_\mu u_\nu - D_\nu u_\mu)$, and the following equation which governs the evolution of the scalar expansion was obtained:

$$\frac{d\theta}{d\lambda} + \frac{1}{3} \theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} - 2 h^{\mu\nu} D_\mu a_\nu + R_{\mu\nu} u^\mu u^\nu = 0. \quad (5)$$

Before discussing the correctness of this QRE presented in Eq. (5), let us mention few important remarks and arguable points about the whole approach which can be summarized as follows:

- (i) The norm of the tangent field u^μ , as found in Eq.(3), does not have a well defined sign. The expected deviation from $(-c^2)$ is considered to be small, but there seems to be no control on the magnitude of this deviation, especially when the curvature becomes large, let alone the unknown sign of the term $\frac{\hbar^2}{m^2} \frac{\square \mathcal{R}}{\mathcal{R}}$. Therefore, the points along the trajectory can not be chronologically ordered in a unique way which is independent of any reference frame. This violation of causality can not be reconciled in any reasonable manner. Although nonrelativistic quantum mechanics allows for causality violation, while relativistic theory softens this violation, the quantum field theory completely evades the causality violation in a miraculous and well-known way [2, 3] through the exact cancelation between the contribution of particles and that of antiparticles. This serious flaw was pointed out in [4] concerning the cosmological study based on QRE carried out in [5].
- (ii) The projector $h_{\mu\nu}$ is not properly defined in terms of normalized tangent vector \hat{u}^μ ; more precisely it cannot be taken as a spatial metric nor $h^\mu{}_\nu$ is a projector onto the subspace of tangent space perpendicular to u^μ .
- (iii) The resulting tensor $D_\nu u_\mu$ is no longer purely spatial unless the congruence generated by u^μ is a timelike geodesic.
- (iv) The expansion scalar θ will no longer have the same interpretation as, for instance, a measure of the fractional rate at which the volume of a ball of matter changes with respect to time as measured by a central comoving observer.
- (v) The tensor $\sigma_{\mu\nu}$ is not any longer purely spatial and consequently $\sigma_{\mu\nu} \sigma^{\mu\nu}$ will not have a fixed signature; therefore, the positivity energy conditions cannot be used without considering the signature of this term.

It thus results that the whole approach is ill defined and loses a considerable part of its geometrical and physical meaning.

As to the correctness of the modified Raychaudhuri equation (QRE) derived in [1], i.e., Eq. (5), and apart from the factor 2, which is related to the factor 1/2 mentioned before, it seems that the author of [1] took only into account the modification arising from $D_\mu a^\mu$, which is non vanishing by virtue of Eq. (4). However it should be mentioned that not all the results obtained for geodesic congruence nor the formalism carry over directly to the case of congruence generated by timelike curves. Indeed, there are many other terms coming from the facts that $D_\rho h_{\mu\nu} \neq 0$, $h^\mu{}_\nu$ is no longer projector nor $h_{\mu\nu}$ has trace equals to 3, and $\sigma_{\mu\nu} h^{\mu\nu} \neq 0$. Now, even though we find the whole formalism used in [1] ill defined, it would be interesting for the sake of completeness that we give the correct “formal” equation that would arise by taking into accounts all the terms. The resulting modified Raychaudhuri equation turns out to be

$$\frac{d\theta}{d\lambda} - a_\nu a^\nu - \frac{1}{9} \theta^2 h_{\mu\nu} h^{\mu\nu} + \frac{2}{3} \theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} - h^{\mu\nu} D_\mu a_\nu + R_{\mu\nu} u^\mu u^\nu = 0. \quad (6)$$

Comparing this finding in Eq. (6) with that of Eq. (5), we find that the two equations are completely different with the exception of the last three terms.

Although the mathematical approach devised in [1] leads to incorrect modified Raychaudhuri equation, there is still a correct modified equation which can be deduced for the Bohmian approach proposed therein; of course we leave aside the problems with Bohm's interpretation and its relativistic generalization.

It turns out that there are two equivalent formalisms which can be used to write a modified Raychaudhuri equation for the congruence generated by the velocity field satisfying Eqs. (2) and (3). The first approach is to use directly the non-normalized velocity field (assuming that it remains timelike) as given by Eqs.(2) and (3) and define all mathematical quantities properly. This approach is slightly different from the standard approach and thus one has to derive the corresponding Raychaudhuri equation from scratch. The proper spatial metric can be defined as

$$h_{\mu\nu} = g_{\mu\nu} - \frac{u_\mu u_\nu}{u_\alpha u^\alpha}. \quad (7)$$

The expansion tensor is defined by projecting the symmetric part of $D_\mu u_\nu$ onto the orthogonal space (in our case $D_\mu u_\nu$ is already symmetric),

$$\Theta_{\mu\nu} = h_\mu^\alpha h_\nu^\rho D_\alpha u_\rho. \quad (8)$$

The resulting expansion tensor can be seen to be purely spatial, i.e., $\Theta_{\mu\nu} u^\mu = \Theta_{\mu\nu} u^\nu = 0$, and the expansion scalar can then be defined as $\Theta = h^{\mu\nu} \Theta_{\mu\nu} = g^{\mu\nu} \Theta_{\mu\nu}$. The expansion tensor $\Theta_{\mu\nu}$ can be decomposed in terms of its irreducible parts as

$$\Theta_{\mu\nu} = \sigma_{\mu\nu} + \frac{\Theta}{3} h_{\mu\nu}, \quad \text{where, } \sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{\Theta}{3} h_{\mu\nu}. \quad (9)$$

It is straightforward to show that the scalar expansion satisfies the following modified Raychaudhuri equation (or QRE):

$$\frac{d\Theta}{d\lambda} + \frac{1}{3} \Theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} - h^{\mu\nu} D_\mu a_\nu + 3 h^{\mu\nu} \frac{a_\mu a_\nu}{u^2} + R_{\mu\nu} u^\mu u^\nu = 0. \quad (10)$$

Except for the term $h^{\mu\nu} D_\mu a_\nu$ (up to a factor 2) our equation is substantially different from the equation obtained in [1], i.e., Eq. (5). Moreover the tensor $\sigma_{\mu\nu}$ and the projector h_μ^ν are very different from the ones given in [1]. Actually it is the fact that $\sigma_{\mu\nu}$ is being purely spatial which makes the positive energy conditions of particular importance.

The second alternative approach to derive a modified Raychaudhuri equation for the velocity field u^μ is to normalize it viz $\hat{u}_\mu = \frac{D_\mu S}{\sqrt{|D_\alpha S D^\alpha S|}}$. Having a normalized timelike velocity field \hat{u}_μ together with all necessary derived quantities such as spacial projector $h_{\mu\nu}$, θ (expansion), $\sigma_{\nu\mu}$ (shear tensor), $\omega_{\nu\mu}$ (rotation tensor) and a^μ (acceleration) which are defined as

$$\left. \begin{aligned} h_{\mu\nu} &= g_{\mu\nu} + \hat{u}_\mu \hat{u}_\nu, \quad \text{where } \hat{u}_\alpha \hat{u}^\alpha = -1, \\ \theta &= D_\mu \hat{u}^\mu, \\ \sigma_{\nu\mu} &= \frac{1}{2} h_\nu^\rho h_\mu^\sigma (D_\rho \hat{u}_\sigma + D_\sigma \hat{u}_\rho) - \frac{1}{3} \theta h_{\mu\nu}, \\ \omega_{\nu\mu} &= \frac{1}{2} h_\nu^\rho h_\mu^\sigma (D_\rho \hat{u}_\sigma - D_\sigma \hat{u}_\rho), \\ a^\mu &= \hat{u}^\rho D_\rho \hat{u}^\mu. \end{aligned} \right\} \quad (11)$$

It is straightforward to use the standard formalism devised for a generic timelike curve as described in [6, 7] and derive the following corrected or modified Raychaudhuri equation that governs the evolution of the expansion parameter θ as

$$\frac{d\theta}{d\lambda} + \frac{1}{3} \theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} - D_\mu a^\mu + R_{\mu\nu} \hat{u}^\mu \hat{u}^\nu = 0. \quad (12)$$

This equation is again completely different from Eq. (5). As can be seen, the rotation tensor is no longer vanishing and the other quantities differs in their definition from the ones given in [1].

Another incidence which we see presenting a problem in the approach of QRE is the one related to Jacobi equation. By using the standard definition for the relative acceleration between geodesics and applying it to the case of nongeodesic timelike curves, where the term $D_\beta (u^\gamma D_\gamma u^\mu)$ is not vanishing, one gets the modified Jacobi equation,

$$\begin{aligned}\frac{D^2 \eta^\mu}{d\lambda^2} &\equiv u^\gamma D_\gamma (u^\beta D_\beta \eta^\mu), \\ &= \eta^\beta D_\beta (u^\gamma D_\gamma u^\mu) - R^\mu_{\alpha\beta\gamma} u^\alpha u^\gamma \eta^\beta, \\ &= -R^\mu_{\alpha\beta\gamma} u^\alpha u^\gamma \eta^\beta + \frac{\hbar^2}{2m^2} \eta^\beta D_\beta D^\mu \left\{ \left(\frac{\square \mathcal{R}}{\mathcal{R}} \right) + \epsilon R \right\}.\end{aligned}\quad (13)$$

Again, this is to be compared to the different result in [1] (after setting $\epsilon = 0$) :

$$\frac{D^2 \eta^\mu}{d\lambda^2} = -R^\mu_{\alpha\beta\gamma} u^\alpha u^\gamma \eta^\beta + \frac{\hbar^2}{m^2} \eta^\gamma D_\gamma D^\mu \left(\frac{\square \mathcal{R}}{\mathcal{R}} \right). \quad (14)$$

We again note that both equations, Eq. (13) and Eq. (14), do not correspond correctly to the deviation of timelike curves for reasons similar to the ones we mentioned for the Raychaudhuri equation. The author of [1] used the same definition for the relative acceleration between geodesics and took only into consideration the nonvanishing of $D_\beta (u^\gamma D_\gamma u^\mu)$; however, to properly derive the Jacobi equation for timelike curves, the deviation vector η^μ has to be defined in a more subtle way by considering the relative separation η^μ modulo a component parallel to \hat{u}^ν and thus the projection of η^μ i.e $\eta^\mu_\perp = h^\mu_\nu \eta^\nu$. This subtlety does not arise in case of geodesic congruence because η^μ and \hat{u}^ν can always be chosen to be orthogonal along the geodesic. For affinely parameterized timelike curves with parameter λ , the deviation equation, after lengthy algebraic manipulations, is given by [6]

$$\begin{aligned}h^\mu_\alpha \frac{D}{d\lambda} (h^\alpha_\beta \frac{D}{d\lambda} \eta^\beta_\perp) &\equiv h^\mu_\alpha \hat{u}^\delta D_\delta \left[h^\alpha_\beta (\hat{u}^\gamma D_\gamma \eta^\beta_\perp) \right], \\ &= -R^\mu_{\alpha\beta\gamma} \eta^\beta_\perp \hat{u}^\alpha \hat{u}^\gamma + h^\mu_\alpha (D_\beta a^\alpha) \eta^\beta_\perp + a^\mu a_\alpha \eta^\alpha_\perp,\end{aligned}\quad (15)$$

where a_μ is the acceleration as defined in Eq. (11). It is obvious that if one considers timelike curves with non normalized tangent vector ($u^\alpha u_\alpha \neq -1$) then the deviation Eq. (15) would have extra terms on the right hand side. Clearly, using non-normalized velocity fields introduces unnecessary complications without any benefit.

Let us finally comment on the physical conclusions drawn by the author of [1], leaving aside the problems with Bohm's interpretation, the correctness of the modified Raychaudhuri equation and the fact that the formalism itself is ill defined. The claim was that the modification brought to Raychaudhuri equation naturally prevents focussing and the formation of conjugate points. The main argument was the presence of a quantum potential and the fact that congruence was determined through first order differential equation $\left(\frac{dx^\mu}{d\lambda} = \frac{D^\mu S}{m} \right)$ and so the uniqueness would prevent caustic points to exist.

It is true that quantal trajectories available for a single particles in nonrelativistic Bohmian mechanics form a congruence due to the fact that their velocity field is given by ∇S (which is a single-valued function), but this by no means ensures that they will preserve this property (maintain congruence) in the presence of gravity. Indeed this what Raychaudhuri equation and singularity theorem are all about. A congruence in an open region of spacetime is by definition a family of curves such that through each point in this region there passes one and only one curve from this family. The singularity theorem that is generally proven, as presented in [8], starts with a congruence, in particular with a rotationless congruence (or hypersurface orthogonal) and therefore with a velocity field given by $u^\mu = D^\mu f$ (Frobenius theorem), and under some energy positivity conditions it is shown that this family of curves will after a finite proper time fail to be congruence and geodesics get focused as a result of gravity attraction. This is shown by studying Raychaudhuri equation which govern the development of the congruence.

Therefore in order to conclude that focusing never happen within a finite proper time one must follow the development of the velocity field using the new Raychaudhuri equation, by analyzing the different competing terms appearing in the equation before any conclusion can be drawn. This has never been done by the author. Similar remarks applies for Jacobi equation.

Going further now and assuming that there are actually no caustic points for those Bohmian trajectories, this would not change the situation for timelike or null geodesic completeness of the given spacetime. The reason that timelike or null geodesics are important is that the first ones may correspond to a physical observer, while the second ones correspond to a ray of light. In fact, one can extend the notion of completeness to a general timelike curve which can also represent a physical observer, and this extension turns out to be necessary since there are spaces which are geodesically complete but incomplete for timelike curves [9].

On the other hand, it is intriguing to note that the equations deduced for Bohmian trajectories are just for test particles moving under the action of presumed quantum force while keeping the underlying spacetime geometry untouched. This alone can't evade singularity since, for instance, behind black hole horizon neither physical matter nor physical light can escape singularity. Now, even if the conclusions of the author are accepted as they stand, it must be mentioned that such quantum effects should be expected to affect the convergence of timelike geodesics only at very small distances. In such regions the curvature becomes so extreme that it might well count as a singularity; however, it is generally believed that general relativity itself will break down before the final singularity is reached.

To sum up, the QRE approach presented in [1] does not evade singularity, and the whole formulation is, in our opinion, troublesome on both mathematical and physical sides. This certainly invalidates the whole subsequent works [5, 10–14] that adopted the QRE approach and built on it. We believe that this QRE question needs to be fully addressed and settled before any further works based on QRE are contemplated. We hope that the present work will draw attention to this issue, which, somehow, is important in the arena of quantum general relativity.

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